Table of Contents

[Lecture 1 1](#_Toc65425741)

[Lecture 2 - 4 2](#_Toc65425742)

[Lecture 5 2](#_Toc65425743)

[Lecture 6 2](#_Toc65425744)

[Lecture 7 2](#_Toc65425745)

[Lecture 8-9 3](#_Toc65425746)

[Lecture 10 3](#_Toc65425747)

[Lecture 15 3](#_Toc65425748)

[Lecture 16 4](#_Toc65425749)

[Lecture 17 Orthogonal Matrix 4](#_Toc65425750)

[Lecture 18 Determinant 5](#_Toc65425751)

[Lecture 20 5](#_Toc65425752)

[Lecture 21 Eigenvalue and Eigenvectors 6](#_Toc65425753)

[Lecture 22 Diagonalization 6](#_Toc65425754)

[Lecture 23 Differential Equations 7](#_Toc65425755)

[Lecture 24 Markov Matrix and Fourier Series 7](#_Toc65425756)

[Lecture 24B Review 2 8](#_Toc65425757)

[Lecture 25 Symmetric Matrix, Positive Definite 8](#_Toc65425758)

[Lecture 26 Complex matrix, Fast Fourier Transform 9](#_Toc65425759)

[Lecture 27 Positive Definite Matrix 9](#_Toc65425760)

[Lecture 28 Similar Matrix, Jordan Form 9](#_Toc65425761)

[Lecture 29 Singular Value Decomposition 10](#_Toc65425762)

[Lecture 30 Linear Transformation 11](#_Toc65425763)

[Lecture 33 Left Inverse, Right Inverse, Pseudo Inverse 11](#_Toc65425764)

# Lecture 1

1. Linear algebra aims to solve linear equation systems. For example,

can be written as

Therefore, matrix times column vector is a linear combination of columns of A.

Similarly, row vector times matrix is a linear combination of rows of A.

# Lecture 2 - 4

1. We solve linear equation systems by row elimination to convert A to an upper triangle matrix U.

Row elimination (row operation) is equivalent to left multiply by a n elimination matrix E.

Notice that:

1. is lower triangle with only 1 non-zero element at position, which subtracts some multiples of row j from row I, and makes entry of A zero.
2. To calculate , just flip the off-diagonal numbers. is also an elimination matrix.
3. Row elimination is essentially LU decomposition
4. Sometimes row exchange is required in the process of LU decomposition, then becomes

Permutation matrix is the identity matrix with reversed.

1. is invertible is equivalent to:
   1. A has n pivots after elimination (row exchange allowed)
2. Diagonally dominant matrices are invertible. Diagonally dominant matrices are defined as
3. Inverse of upper triangle is upper triangle; Inverse of lower triangle is lower triangle.

# Lecture 5

1. Subspace: a set with components closed under addition and scalar multiplication.
2. Possible subspaces in
3. Column space: all linear combination of columns of A, denoted as
4. Null space:

# Lecture 6

1. can be solvable when
2. The solution of is

# Lecture 7

1. Reduced row echelon form: make all pivot elements 1.

# Lecture 8-9

1. For a matrix, we have.
   * Full column rank: r=n. No free variables. All columns intendent.

has either no solution or unique solution.

* + Full row rank:

always solvable, either solutions

* + Full rank: is invertible,

always has unique solution



# Lecture 10

1. Four basic subspaces

|  |  |  |  |
| --- | --- | --- | --- |
|  | Which space is it in? |  |  |
|  |  |  | First r rows of rref |
|  |  |  | Special solutions |
|  |  |  | Pivot columns |
|  |  |  | Last (m-r) rows of E, where EA=R |

1. Orthogonal subspaces:
2. Orthogonal complement of subspace V contains every vector perpendicular to V.
3. Every vector can be split into a row space component and a null space component.

The basis for row space + basis for null space span

# Lecture 15

1. Project a vector onto
2. Projection & least square: sometimes may have no solution, so solve
3. Projection onto a plane spanned by

# Lecture 16

1. Projection matrix

# Lecture 17 Orthogonal Matrix

1. Vectors are orthonormal if .

, then

1. Examples of orthogonal matrix:
   1. Permutation matrix:
   2. Rotation matrix:
   3. Reflection matrix:
2. Why orthogonal matrix better than ordinary matrix?
   1. Projection matrix
   2. OLS:
3. Gram-Schmidt: convert a set of independent vectors to a set of orthonormal vectors

(remember is row elimination)

1. Any matrix with independent columns can do QR decomposition.

# Lecture 18 Determinant

We use 3 properties to define determinant:

* 1. =

Using the above 3 properties, we can prove more properties.



Proof: from property 2 and 5, if we do LU decomposition then Therefore,

1. Proof is a little tricky. Convert A=LU (if row exchange required then and remaining is same), then. Here we use the fact that

# Lecture 20

Note: Cramer’s rule is computationally expensive and thus not used in practice.

1. Cross product of and is
2. Determinant and area/volume
   1. In 2-D plane, if a parallelogram has two edges then the area is abs of
   2. In 3-D space, if a parallelepiped has three edges , then the volume is abs of
3. Triple product:

# Lecture 21 Eigenvalue and Eigenvectors

1. Eigenvalue and eigenvector for special matrices
   1. Triangle matrix: eigenvalues = diagonal
   2. Singular matrix: is an eigenvalue
   3. Projection matrix: for any vector in plane, ; for any vector plane, (if such vector exists)
   4. Markov matrix (each column adds to 1): is an eigenvalue. (proof: A-I is singular)
3. A and B share the same n independent eigenvectors if and only if AB = BA

# Lecture 22 Diagonalization

1. Assumeis anmatrix, witheigenvectors Then we can write

whereis theeigenvector matrix,is theeigenvalue diagonal matrix.

1. Ifhas independent eigenvectors, then is invertible, andis diagonalizable.
2. haseigenvalues if we count the multiplicity.
3. If allare different, thenis sure to have independent eigenvectors and is diagonalizable.

If there are repeated then we need to consider the algebraic multiplicity (AM: repetitions of ) and geometric multiplicity (GM: number of independent eigenvectors for ).

* If GM=AM for all , then is diagonalizable.
* If GM<AM for some , then is not diagonalizable.

1. There is no connection between invertibility and diagonalizability.

* Invertible:
* Diagonalizable:

1. Calculate for some diagonalizable

Let , then

1. Fibonacci:

2nd order difference convert to 1st order with vector

Let , then

# Lecture 23 Differential Equations

1. Solve where A is a constant matrix. For simplicity we only consider when A is diagonalizable.
2. What is ?

For diagonal matrix ,

For

1. From

we know:

* has the same eigenvector matrix as A
* The eigenvalues of are exponential of eigenvalue of A

# Lecture 24 Markov Matrix and Fourier Series

1. Markov matrix: (1) all entries ; (2) all columns add to 1
2. Properties of a Markov matrix:
   1. is an eigenvalue. Proof:
   2. The eigenvector for has all components
   3. all other eigenvalues. Proof of : (1) have the same eigenvalues. (2) If then Thus for the element with largest absolute value, . Therefore, |
3. Orthonormal basis
4. Fourier series
   1. and are orthogonal
   2. Basis:
   3. Inner product of vectors:

Inner product of functions should be like

* 1. similarly,

We need to divide by to make norm 1

# Lecture 24B Review 2

1. Calculate the determinant of the matrix with

Hint: prove

1. Calculate the determinant of matrix with

Hint: prove

# Lecture 25 Symmetric Matrix, Positive Definite

1. Real symmetric matrix has following 2 properties:
   1. All eigenvalues are real
   2. The eigenvectors can be chosen orthonormal (and has n eigenvectors and thus diagonalizable)
2. Proof of properties in 1:
   1. Proof of 1a: If is complex, we have , then . Since A is real, so Then . As , then . Thus so . As therefore , and is real.
   2. Proof of 1b:

When no repeated and thus , therefore

When there are repeated see section 7.2, or google the proof.

1. Usual case diagonalization:

Symmetric case: , spectral theorem

Every symmetric matrix is a combination of perpendicular projection matrices。

1. For complex matrix, if , then A has real eigenvalues and perpendicular eigenvectors. (Proof: see Point 2. The proof follows when we have for complex matrix).
2. For real symmetric matrix, the number of positive pivots = number of positive eigenvalues
3. Positive definite matric has the following equivalent properties:
   1. all
   2. all pivots f
   3. all sub-determinants

# Lecture 26 Complex matrix, Fast Fourier Transform

1. Hermitian:
2. Complex vs real

|  |  |  |  |
| --- | --- | --- | --- |
| Complex matrix | | Real matrix | |
| Conjugate transpose |  | Transpose |  |
| Hermitian matrix |  | Symmetric matrix |  |
| Inner product |  | Inner product |  |
| Length |  | Length |  |
| Perpendicular |  | Perpendicular |  |
| Unitary Matrix |  | Orthogonal matrix |  |

1. Fourier matrix
2. is orthogonal matrix
3. Fast Fourier Transform

The operation is , but fast Fourier transform is

# Lecture 27 Positive Definite Matrix

1. For a symmetric matrix, positive definite is equivalent to the following properties:
2. In calculus, min

In linear algebra, for min matrix of 2nd derivative is positive definite

1. For a positive definite matrix , is an ellipsoid, with :
   1. Eigenvectors are direction of principle axis
   2. Eigenvalues are
2. Some properties of positive definite matrix:
   1. If A is PD, then is also PD. Proof:
   2. If A, B are PD, then is PD. Proof:
   3. If A is a matrix, then is PD if A has independent columns.

# Lecture 28 Similar Matrix, Jordan Form

1. matrix and are similar if
   1. If is diagonalizable, then and are similar
   2. Similar matrices have same eigenvalues and same number of eigenvectors

Proof:

1. If has n distinct eigenvalues, then A is diagonalizable and similar to

If A has repeated eigenvalues, take matrix with repeated as an example, there are 2 families of similar matrices:

* 1. Only 1 matrix: . It has no other similar matrix, because and .
  2. Big “family” similar to Not diagonalizable with only 1 eigenvector.

1. although they have the same eigenvalue and same number of eigenvectors, because they have different Jordan form.
2. Jordan blocks , with the same ei genvalue on diagonal, and 1 on . Each Jordan block has only 1 eigenvector.
3. Every square matrix A is similar to a Jordan matrix
4. Similar matrices have the same Jordan form.

# Lecture 29 Singular Value Decomposition

1. Let , then

Transform from row space to column space.

1. We want to map to , where is orthogonal basis in row basis, is orthogonal basis in column space.
2. If or , we can add null space as the complements of row space, and add let null space as complement of column space.
3. How to find and

so is the orthogonal eigenvectors of

so is the orthogonal eigenvectors of

If and are not full rank, find their complements to complete it.

Eigenvalues of and are equal ()

Note: when choose eigenvectors there can be two directions. Need to match the sign so that holds. (otherwise, may get )

1. is orthogonal basis for row space

is orthogonal basis for column space

is orthogonal basis for null space

is orthogonal basis for left null space

# Lecture 30 Linear Transformation

1. Linear transformation:
2. Every linear transformation can be represented by a matrix A:
3. Rules to find A, given input basis and output basis

gives the 1st column of A

gives the 2nd column of A

# Lecture 33 Left Inverse, Right Inverse, Pseudo Inverse

1. Left inverse: when has independent columns (), also has independent columns (proof see Lecture 16.5). Thus is invertible.
2. Right inverse: when has independent rows (), is invertible.
3. If we put left inverse on the right of , we get projection matrix onto column space of (Lecture 15 & 16)

If we put right inverse on the left of , we get projection matrix onto row space of

1. Lemma: If are both in row space of , then

Proof: suppose , then and thus is in null space of But is also in row space of .

As row space is orthogonal to null space, the only vector in both row space and null space is 0. Therefore . Contradiction. QED.

1. Lemma (33.4) implies that the mapping from row space to column space, is one to one, and the mapping is “invertible”.

In other words, although A is not invertible in mapping , but when the mapping is limited from row space to column space, it is invertible.

This leads us to the discussion of pseudo inverse.

1. How to find pseudo inverse
   1. , where are invertible, and is matrix:
   2. The pseudo inverse is:

Symmetric matrix: eigenvectors can be made perp; like a real number

Skew-symmetric matrix: . Eigenvalues are pure imaginary

Orthogonal matrix: eigenvalues complex with